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Small Intersection Graph of Subsemimodules of a Semimodule

Ahmed H. Alwan[a,*]

[a]Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq

Abstract

Let R be a commutative semiring with identity, and U be a unitary left R-semimodule. The small intersection graph of non-trivial subsemimodules of U, denoted by $\Gamma(U)$, is an undirected simple graph whose vertices are in one-to-one correspondence with all non-trivial subsemimodules of U and two distinct vertices are adjacent if and only if the intersection of corresponding subsemimodules is a small subsemimodule of U. In this article, we investigate connections between the graph-theoretic properties of $\Gamma(U)$ and some algebraic properties of semimodules. We determine the diameter and the girth of $\Gamma(U)$. We obtain some results for connectivity and planarity of these graphs. Moreover, it is shown that the domination number of a small intersection graph of a semimodule is 1 , whenever U is a subtractive semimodule and direct sum of two simple semimodules.

Keywords: Semimodule, small subsemimodule, small intersection graph, Connectivity, domination number, planarity.

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1. Introduction

Bosak in 1964 [8] introduced the idea of the intersection graph of semigroups. Several other classes of graphs related with algebraic constructions have been also actively examined. For instance, see ([4],[5],[6],[9]). In 2012, the authors in [1] have recognized the intersection graph of submodules of a module. In 2021, the small intersection graph of submodules of a module has been studied by Mahdavi and Talebi in [13]. In this paper, we introduced the small intersection graph of subsemimodules of a semimodule. Our chief goal is to study the joining among the algebraic properties of semimodules and the graph theoretic properties of the graph related with it.

In Section 2, we show that $\Gamma(U)$ is complete if either U is a subtractive semimodule and direct sum of two simple semimodules or U is hollow semimodule. We proved that if U be a subtractive semisimple semimodule such that it is not simple, then $\operatorname{diam}(\Gamma(U)) \leqslant 3$. We establish that if $|S(U)| \in \{1, 2\}$ and under some condition, then $\Gamma(U)$ is a planar graph. Moreover, if $|S(U)| \geqslant 3$, then $\Gamma(U)$ is not a planar graph. In Section 3, we show that if $U = N \oplus L$ is a subtractive semimodule, where N and L are two simple semimodules, then $\gamma(\Gamma(U)) = 1$.

Throughout this paper R is a commutative semiring with identity and U is a unitary left R semimodule. A commutative semiring R is defined as an algebraic structure $(R, +, \cdot)$ where $(R, +)$ and $(R, \cdot)$ are commutative semigroups, joined by $a(b + c) = ab + ac$ for all a, b and c of R and there exist $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ and $1r = r$ for all r of R see ([2], [3], and [11]).

*Corresponding author

Email address: ahmedha_math@utq.edu.iq (Ahmed H. Alwan)

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Let $(U, +)$ be an additive abelian monoid with additive identity 0 , then U is a left R-semimodule if there exists a scalar multiplication $R \times U \to U$ denoted by $(r, u) \to ru$, such that $(rr')u = r(r'u)$; $r(u + u') = ru + ru'$; $(r + r')u = ru + r'u$; and $0u = r0_U = 0_U$ for all $r, r' \in R$ and all $u, u' \in U$. If the condition $1u = u$ for all $u \in U$ hold then the semimodule U is said to be unitary. A subset N of an R-semimodule U is called a subsemimodule of U, we write $N \leqslant U$ (or $N \subseteq U$), if for $n, n' \in N$ and $r \in R, n + n' \in N$ and $rn \in N$. A subsemimodule N is called a subtractive subsemimodule (or k subsemimodule) of U if $x, x + y \in N$, then $y \in N$ [11]. We say an R-semimodule is subtractive if each of its R-subsemimodules is subtractive. We mean from a non-trivial subsemimodule of U is a non-zero proper subsemimodule of U. A subsemimodule N of an R-module U is called small (= superfluous) in U (we write $N \ll U$). if for every subsemimodule $L \leqslant U$, with $N + L = U$ implies that $L = U$ [14]. A nonzero R-semimodule U is called hollow, if every proper subsemimodule of U is small in U. The semimodule U is called simple if it has no proper subsemimodules, and U is said to be semisimple if it is a direct sum of simple subsemimodules. A subsemimodule M of a semimodule U is maximal if and only if it is not properly contained in any other subsemimodule of U. An R-semimodule U is said to be local if it has a unique maximal subsemimodule M and we denote it by (U, M). The set of maximal subsemimodules of U is denoted by $\max(U)$. The Jacobson radical of an R-semimodule U, denoted by $\mathrm{Rad}(U)$, is the sum of all small subsemimodules of U, and also, is the intersection of all maximal subsemimodules of U. The socle of a semimodule U, denoted by $\mathrm{Soc}(U)$, is the sum of all simple subsemimodules of U. The reference for semimodule theory is [11]; for graph theory is [7].

Let Γ be a graph with the vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The order of Γ is the number of vertices of Γ and we denoted it by $|\Gamma|$. A graph Γ is finite, if $|\Gamma| < \infty$, otherwise, Γ is infinite. If u and v are two adjacent vertices of Γ, then we write $u - v$. The degree of a vertex v in a graph Γ, denoted by $\deg(v)$, is the number of edges incident with v. The minimum degree of Γ is $\delta(\Gamma)$. Let u and v be two distinct vertices of Γ. An u, v-path is a path with starting vertex u and ending vertex v. For distinct vertices u and v, $d(u, v)$ is the least length of an $u, v-$ path. If Γ has no such a path, then $d(u, v) = \infty$. The diameter of Γ, denoted by $\mathrm{diam}(\Gamma)$, is the supremum of the set $\{d(x, y) : u \text{ and } v \text{ are distinct vertices of } \Gamma\}$. A cycle in a graph is a path of length at least 3 through different vertices which begins and ends at the same vertex. The girth of a graph Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. A graph is called connected, if there is a path between every pair of vertices of the graph. A tree is a connected graph which does not contain a cycle. A star graph is a tree consisting of one vertex adjacent to all the others. A complete graph with n distinct vertices, denoted by K_n. A complete bipartite graph with two part sizes m and n is denoted by $K_{m,n}$. By a clique in a graph Γ, we mean a complete subgraph of Γ.

2. Connectivity, Diameter And Girth of $\Gamma(U)$

In this section, we characterize some semimodules whose small intersection graphs of nontrivial subsemimodules are connected and complete. Also, the diameter and the girth of $\Gamma(U)$ are determined.

Definition 2.1. The small intersection graph of non-trivial subsemimodules of an R-semimodule U, denoted by $\Gamma(U)$, is an undirected simple graph whose vertices are in one-to-one correspondence with all non-trivial subsemimodules of U and two distinct vertices N and L are adjacent if and only if $N \cap L \ll U$.

Proposition 2.2. Let U be an R-semimodule with the graph $\Gamma(U)$. Then $\Gamma(U)$ is complete, if one of the following holds.

(1) If $U = U_1 \oplus U_2$ is a subtractive semimodule, where U_1 and U_2 are two simple R-semimodules.

(2) If U is hollow.

Proof. (1) Let a left subtractive R-semimodule U be a sum $U = U_1 \oplus U_2$ such that U_1 and U_2 are two simple R-semimodules. So by [12, Theorem 3.10], $U_1 + U_2 = U$ and $U_1 \cap U_2 = \{0\}$. Then every nontrivial

subsemimodule of U is simple. Let N and L be two distinct vertices of the graph $\Gamma(U)$. Then they are the non-trivial subsemimodules of U which are simple and minimal. Furthermore, $N \cap L \subseteq N, L$ and if $N \cap L \neq (0)$, then minimality of N and L implies that $N \cap L = N = L$, which is a contradiction. Thus, $N \cap L = (0) \ll U$, hence $\Gamma(U)$ is a complete graph.
(2) Let U be a hollow R-semimodule. Suppose that N_1 and N_1 are two distinct vertices of the graph $\Gamma(U)$. Hence N_1 and N_1 are two nonzero small subsemimodules of U. As $N_1 \cap N_2 \subseteq N_i$, for $i = 1, 2$, by [14, Proposition 3], $N_1 \cap N_2 \ll U$, hence $\Gamma(U)$ is a complete graph. □

By Part 2 of Proposition 2.2, we have the following corollary.

Corollary 2.3. Let U be an R-semimodule. Then the following hold:

(1) If U is a local semimodule, then the graph $\Gamma(U)$ is complete.

(2) Every nonzero small subsemimodule of U is adjacent to all other vertices of $\Gamma(U)$ and the induced subgraphs on the sets of small subsemimodules of U are cliques.

(3) If the subsemimodules of U form a chain, then the graph $\Gamma(U)$ is complete.

Proof. (1) Suppose that U is a uniserial R-semimodule. Then each two nontrivial subsemimodules of U are comparable. Clearly, every nontrivial subsemimodule of U is a small subsemimodule. Hence, U is a hollow R-semimodule. So, by Proposition 2.2(2), $\Gamma(U)$ is complete. Also, if U is a local R-semimodule, then local R-semimodules are hollow. Again, by Proposition 2.2(2), $\Gamma(U)$ is complete.
(2) Evident.
(3) If the subsemimodules of U form a chain, then it is easy to check that any two vertices of $\Gamma(U)$ are adjacent. □

Example 2.1. Assume $\mathbb{Z}_0 = \mathbb{Z}^+ \cup \{0\}$ is the semiring of non-negative integers, then the $\mathbb{Z}_0$-semimodule $\mathbb{Z}_0$ is local with maximal subsemimodule $N = \mathbb{Z}_0 \backslash \{1\}$, see [11], So, it is hollow. Also, For every prime number p and for all $n \in \mathbb{Z}^+$with $n \geqslant 2$, the $\mathbb{Z}$-semimodule $\mathbb{Z}_{p^n}$ is local, then it is hollow. Also, since any two subsemimodules of $\mathbb{Z}$-semimodule $\mathbb{Z}_{p^\infty}$ are comparable, then every proper subsemimodule of $\mathbb{Z}_{p^\infty}$ is small in $\mathbb{Z}_{p^\infty}$. Hence for every prime number p, the $\mathbb{Z}$-semimodule $\mathbb{Z}_{p^\infty}$ is hollow. By Proposition 2.2 (2), $\Gamma(\mathbb{Z}_0), \Gamma(\mathbb{Z}_{p^n})$ and $\Gamma(\mathbb{Z}_{p^\infty})$ are complete graphs.

Example 2.2. Consider the semiring $R = \{0, 1, u\}$ with the following addition and multiplication:

(a) $0+0=0, 0+1=1, 0+u=u, 1+1=1, 1+u=1, u+u=u$.

(b) $0 \times 0=0, 0 \times 1=0, 0 \times u=0, 1 \times 1=1, 1 \times u=u, u \times u=u$.

R is a commutative semiring with identity 1 . Also, $\{0, u\}$ is a unique nontrivial subsemimodule of the R-semimodule R. Thus $|\Gamma({}_R R)| = 1$, and $\Gamma({}_R R) \cong K_1$.

Example 2.3. Set $R = B(p^h, 0) = \{0, 1, \cdots, p^h - 1\}$, where p is a prime integer and $h \in \mathbb{Z}^+$and define an operation $\oplus$ on R as follows: If $a, b \in R$ then $a \oplus b = a + b$ if $a + b \leqslant p^h - 1$ and, otherwise, $a \oplus b$ is the unique element c of R satisfying $c \equiv a + b \pmod{p^h}$. Define the an operation $\odot$ on R similarly. Then, $(R, \oplus, \odot)$ is a local semiring, see [11, Example 6.1]. So, the graph of the R-semimodule R is complete, by Part 2 of Corollary 2.3.

Theorem 2.4. [12, Theorem 3.10] let U be a subtractive R-semimodule. Then U is semisimple and each if and only if for every subsemimodule $N \subseteq M$, there exists a subsemimodule $K \subseteq M$ such that $M = N + K$ and $N \cap K = 0$.

In [10.2.8(9)], the result was proved for modules. we will prove it for semimodules.

Lemma 2.5. Let U be a subtractive R-semimodule. Then $\operatorname{Soc}(\operatorname{Rad}(U)) \ll U$.

Proof. Put $N = \text{Soc}(\text{Rad}(U))$ and assume $U = N + K$ for some subsemimodule $K \subseteq U$. Setting $V = N \cap K$, we have $N = V \oplus V'$ for some $V' \subseteq N$ and $U = N + K = (V \oplus V') + K = V' \oplus K$, by Theorem 2.4. Now any simple subsemimodule of V' is a direct summand of (V' and) U and is small in U and thus is zero. Hence, $V' = 0$ and so $K = U$, therefore $N \ll U$. □

Lemma 2.6. (1) Let U be a subtractive semimodule and N be a finitely generated subsemimodule of U which is contained in $\text{Rad}(U)$. Then $N \ll U$.

(2) Let U be a subtractive semimodule and N be a semisimple subsemimodule of U which is contained in $\text{Rad}(U)$. Then $N \ll U$.

Proof. (1) Suppose that $N \leqslant U$ is finitely generated. Then, $N = Rn_1 + Rn_2 + \cdots + Rn_r$ for some $n_i \in N$, $1 \leqslant i \leqslant r$. By [14], $Rn_i \ll U$, since U is subtractive and $Rn_i \subseteq \text{Rad}(U)$. By [14, Proposition 3(b)], $N \ll U$.
(2) Suppose that N is a semisimple subsemimodule of U. Then $\text{Soc}(N) = N$ and since $N \subseteq \text{Rad}(U)$, $\text{Soc}(N) \subseteq \text{Soc}(\text{Rad}(U))$. Also, by Lemma 2.5, $\text{Soc}(\text{Rad}(U)) \ll U$. Thus by [14, Proposition 3(b)], $N \ll U$. □

Proposition 2.7. [11, Proposition 14.22] (Semimodularity Law) Let U be a semimodule over semiring R and let N and K be subsemimodules of U. Let L be a k-subsemimodule of U with $N \subseteq L$. Then $L \cap (N + K) = N + (L \cap K)$.

Proposition 2.8. Let U be a subtractive semimodule with the graph $\Gamma(U)$ and $\text{Rad}(U) \neq (0)$. Then the following conditions hold:

(1) If N is a non-trivial subsemimodule of U which is direct summand of U with $(0) \neq \text{Rad}(N) \ll U$, then $\Gamma(U)$ contains at least one cycle of length 3 .

(2) If T is a non-trivial finitely generated or semisimple subsemimodule of U contained in $\text{Rad}(U)$. Then $d(T, \text{Rad}(U)) = 1$ and $d(T, L) = 1$ for every non-trivial subsemimodule L of U.

Proof. (1) Since N is a direct summand of U, by [12, Theorem 3.10], then there exists a subsemimodule K of U such that $N \oplus K = U$. Then $\text{Rad}(N) \oplus \text{Rad}(K) = \text{Rad}(U)$. Since $\text{Rad}(N) \subseteq N$ and $N\cap \text{Rad}(K) \subseteq N \cap K = (0)$, by the semimodularity law (Proposition 2.7), $N \cap \text{Rad}(U) = \text{Rad}(N)$. Then $N \cap \text{Rad}(U) \ll U$. Also, $\text{Rad}(N) = N \cap \text{Rad}(N) \ll U$ and $\text{Rad}(N) = \text{Rad}(N) \cap \text{Rad}(U) \ll U$ and we haved $(N, \text{Rad}(U)) = 1$, $d(N, \text{Rad}(N)) = 1$ and $d(\text{Rad}(N), \text{Rad}(U)) = 1$. Hence, $\{N, \text{Rad}(N), \text{Rad}(U)\}$ is a cycle. Thus, the graph $\Gamma(U)$ contains at least one cycle of length 3.
(2) Suppose that T is a non-trivial finitely generated or semisimple subsemimodule of U. Then by Lemma 2.9, $T \ll U$. Since $T \subseteq \text{Rad}(U), T = T \cap \text{Rad}(U) \ll U$ and since $T \cap L \subseteq T, T \cap L \ll U$ for every other non-trivial subsemimodule L of U. Hence $d(T, \text{Rad}(U)) = 1$ and $d(T, L) = 1$. □

Corollary 2.9. Let U be a subtractive R-semimodule. If U has at least one non-zero small subsemimodule, then $\Gamma(U)$ is a connected graph and diam $(\Gamma(U)) \leqslant 2$.

Corollary 2.10. Let U be a subtractive R-semimodule with $\text{Rad}(U) \neq (0)$. Then $\Gamma(U)$ is a connected graph, if one of the following holds.

(1) The semimodule U is finitely generated.

(2) There exists a non-trivial subsemimodule of U which is finitely generated or semisimple contained in $\text{Rad}(U)$.

Proof. Part 1 is obvious and Part 2 follows from Lemma 2.6(2) and Corollary 2.9. □

Proposition 2.11. Let U be an R-semimodule with graph $\Gamma(U)$. If $\Gamma(U)$ has no isolated vertex, then $\Gamma(U)$ is connected and $\text{diam}(\Gamma(U)) \leqslant 3$.

Proof. Let A and B be two non-adjacent vertices of $\Gamma(U)$. Since $\Gamma(U)$ has no isolated vertex, there exist subsemimodules A_1 and B_1 such that $A \cap A_1 \ll U$ and $B \cap B_1 \ll U$. Now, if $A_1 \cap B_1 \ll U$, then $A - A_1 - B_1 - B$ is a path of length 3. Otherwise $A - A_1 \cap B_1 - B$ is a path of length 2. It follows that $\Gamma(U)$ is a connected graph and diam $(\Gamma(U)) \leqslant 3$. □

Theorem 2.12. Let U be a subtractive semisimple R-semimodule such that it is not simple. Then the following statements hold:

(1) The graph $\Gamma(U)$ has no isolated vertex.

(2) The graph $\Gamma(U)$ is connected and $\mathrm{diam}(\Gamma(U)) \leqslant 3$.

Proof. (1) Let X be a vertex of the graph $\Gamma(U)$. Since U is subtractive semisimple semimodule, then every subsemimodule of U is a direct summand of U by [12, Theorem 3.10]. Thus there exists a subsemimodule Y of U such that $U = X \oplus Y$. Hence $X \cap Y = (0) \ll U$ and thus, there exists an edge between vertex X of $\Gamma(U)$ and another vertex of this graph. Then X is not an isolated vertex. Consequently, $\Gamma(U)$ has no isolated vertex.
(2) By Proposition 2.14 and Part 1. □

In the next example we give a semimodule U such that it is not semisimple and the graph $\Gamma(U)$ is connected and $\mathrm{diam}(\Gamma(U)) = 1$.

Example 2.4. Let $\mathbb{Z}_{18}$ be a $\mathbb{Z}$-semimodule. Then $V(\Gamma(\mathbb{Z}_{18})) = \{\langle\overline{2}\rangle, \langle\overline{3}\rangle, \langle\overline{6}\rangle, \langle\overline{9}\rangle\}$, the graph $\Gamma(\mathbb{Z}_{18})$ is connected and diam $(\Gamma(\mathbb{Z}_{18})) = 1$. See Figure.1

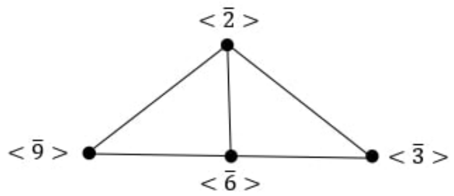

Figure 1: $\Gamma(\mathbb{Z}_{18})$

Example 2.5. Consider $\mathbb{Z}_{24}$ the semiring of integers modulo 24 as a $\mathbb{Z}$-semimodule. Then $V(\Gamma(\mathbb{Z}_{24})) = \{\langle\overline{2}\rangle, \langle\overline{3}\rangle, \langle\overline{6}\rangle,$
$\langle\overline{9}\rangle\}$, the graph $\Gamma(\mathbb{Z}_{24})$ is connected and diam $(\Gamma(\mathbb{Z}_{24})) = 2$. See Figure 2

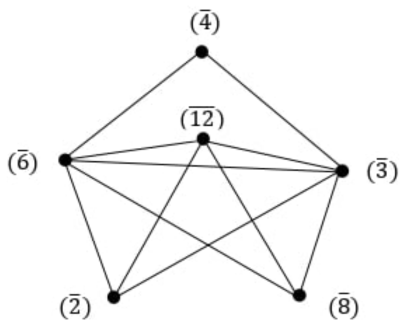

Figure 2: $\Gamma(\mathbb{Z}_{24})$

For a semimodule U we use $S(U)$ which denotes the set of non-zero small subsemimodules of U.

Proposition 2.13. Let U be an R-semimodule and $|\mathbb{S}(U)| \geqslant 1$. If $\Gamma(U)$ is a tree, then $\Gamma(U) = K_1$ or $\Gamma(U)$ is a star graph.

Proof. Assume that $\Gamma(U)$ is a tree. Then $|\mathbb{S}(U)| < 2$. Otherwise, let X and Y be two non-zero small subsemimodules of U. So $(X, X \cap Y, Y)$ is a cycle of length 3 , a contradiction. Since $|\mathbb{S}(U)| \geqslant 1$, then $|\mathbb{S}(U)| = 1$. Hence, U has a unique non-zero small subsemimodule. Let $N \in \mathbb{S}(U)$. For every vertex L of $\Gamma(U)$, if $L = N$, then $\Gamma(U) \cong K_1$ and if $L \neq N$, as $L \cap N \ll U$, we deduce $\Gamma(U) \cong K_2$. Now, let $\Omega = \{v_i \mid v_i \neq N, i \in I\}$. Then every two arbitrary distinct vertices v_i and $v_j, i \neq j$, are not adjacent and for $i \neq j, v_i - N - v_j$ is a path and hence $\Gamma(U)$ is star graph. □

Proposition 2.14. Let U be an R-semimodule with the graph $\Gamma(U)$. If $|\mathbb{S}(U)| \geqslant 2$, then $\Gamma(U)$ contains at least one cycle and $gr(\Gamma(U)) = 3$.

Proof. Suppose that $|\mathbb{S}(U)| \geqslant 2$. Then U has at least two different nonzero small subsemimodules, say N_1 and N_2. Since $N_1 \cap N_2 \subseteq N_i$, for $i = 1, 2$, by [14, Proposition 3(a)], $N_1 \cap N_2 \ll U$. Also, $N_1 \cap (N_1 \cap N_2) \ll U$ and $N_2 \cap (N_1 \cap N_2) \ll U$. We consider two possible cases for $N_1 \cap N_2$.

Case 1: If $N_1 \cap N_2 \neq (0)$, then $d(N_1, N_2) = 1, d(N_1, N_1 \cap N_2) = 1$ and $d(N_2, N_1 \cap N_2) = 1$. Thus $(N_1, N_1 \cap N_2, N_2)$ is a cycle of length 3. Also by [14, Proposition 3(b)], $N_1 + N_2 \ll U$ and since $N_1 \cap (N_1 + N_2) \ll U$ and $N_2 \cap (N_1 + N_2) \ll U$, $(N_1, N_1 + N_2, N_2)$ is a cycle of length 3. Similarly, $(N_1 \cap N_2, N_1, N_1 + N_2)$ and $(N_1 \cap N_2, N_2, N_1 + N_2)$ are cycles of length 3 and $(N_1, (N_1 + N_2), N_2, N_1 \cap N_2, N_1)$ is a cycle of length 4.

Case 2: If $N_1 \cap N_2 = (0)$, then $(N_1, N_1 + N_2, N_2)$ is a cycle of length 3 in the graph $\Gamma(U)$. Therefore, $\Gamma(U)$ contains at least one cycle and so $gr(\Gamma(U)) = 3$. □

3. Domination And Planarity of $\Gamma(U)$

In this section, we conclude the domination of $\Gamma(U)$. Also, we revision the relationship between the planarity of $\Gamma(U)$ and the nonzero small subsemimodules of U.

We recall that for a graph Γ, a subset D of the vertex-set of Γ is called a dominating set (or DS) if every vertex not in D is adjacent to a vertex in D. The domination number, $\gamma(\Gamma)$, of Γ is the minimum cardinality of a dominating set of Γ.

In this paper, a subset D of the vertex set of the graph $\Gamma(U)$ is a DS if and only if for any nontrivial subsemimodule N of U there is a L in D such that $N \cap L \ll U$.

Lemma 3.1. Let U be an R-semimodule with $|\Gamma(U)| \geqslant 2$, then the following hold:

(1) If D is a subset of the vertex set of the graph $\Gamma(U)$ such that D either contains at least one small subsemimodule of U or there exists a vertex $X \in D$ which $X \cap Y = (0)$, for every vertex $Y \in V(\Gamma(U)) \backslash D$. Then D is a DS in $\Gamma(U)$.

(2) If U has at least one nonzero small subsemimodule, then for each nonzero small subsemimodules X of $U, \{X\}$ is a DS and $\gamma(\Gamma(U)) = 1$.

Proposition 3.2. Let $U = N \oplus L$ be a subtractive R-semimodule, where N and L are two simple R-semimodules. Then $\gamma(\Gamma(U)) = 1$.

Proof. Assume $U = N \oplus L$, such that N and L are two simple R-semimodules. By Proposition 2.2 (1), $\Gamma(U)$ is a complete graph. Let X be an arbitrary vertex of the graph $\Gamma(U)$. Then for any distinct vertex Y of $\Gamma(U), X \cap Y \ll U$, thus $\{X\}$ is a DS and $\gamma(\Gamma(U)) = 1$. □

Theorem 3.3. Let U be a R-semimodule with $|\mathbb{S}(U)| \geqslant 2$ and $|\Gamma(U)| \geqslant 3$. Then the following conditions hold:

(1) If I and J are two small subsemimodules of U then there exists $K \in V(\Gamma(U))$ such that $K \in N(I) \cap N(J)$.

(2) The graph $\Gamma(U)$ has at least one triangle.

Proof. It is clear. □

Proposition 3.4. Let U be an R-semimodule. Then the following are equivalent:

(1) The graph $\Gamma(U)$ has no triangle.

(2) If $\{I, J\} \in E(\Gamma(U))$, then there is no $K \in V(\Gamma(U))$ such that $K \in N(I) \cap N(J)$.

(3) The semimodule U has at most one nonzero small subsemimodule such that the intersection of every pair of the non-small nontrivial subsemimodules of U is non-small.

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (3) Assume that for every two adjacent vertices of the graph $\Gamma(U)$, there is no $K \in V(\Gamma(U))$ such that $K \in N(I) \cap N(J)$. Let there exist at least two nonzero small subsemimodules S_1 and S_2 of U. Since $S_1 \cap S_2 \ll U$, they are adjacent vertices of the graph $\Gamma(U)$ and also, there is no $K \in V(\Gamma(U))$ such that $K \in N(I) \cap N(J)$, which is a contradiction by Theorem 3.3(1).
(3) $\Rightarrow$ (1) Assume U has no nonzero small subsemimodule. Since the intersection of every pair of the non-small nontrivial subsemimodules of U is non-small, $\Gamma(U)$ has no triangle. Moreover, Let S be the only nonzero small subsemimodule of U. Then for every three arbitrary vertices N_1, N_2, and N_3 of the graph $\Gamma(U)$, at least two of them are non-small. Let $S = N_1$. As $N_2 \cap N_3$ is non-small subsemimodules of U, then $N_2 - S - N_3$ is a path. Also if $S \neq N_i$, for $i = 1, 2,3$. Since $N_i \cap N_j$ is non-small subsemimodules of U, for $i, j = 1, 2, 3$ and $i \neq j$, then N_1, N_2, and N_3 are not adjacent vertices in the graph $\Gamma(U)$. Hence, there is no triangle in the graph $\Gamma(U)$. □

Proposition 3.5. Let U be a finitely generated subtractive R-semimodule such that $\mathrm{Rad}(U) \neq (0)$, then the graph $\Gamma(U)$ has a triangle.

Proof. Since U is a finitely generated R-semimodule, hence $(0) \neq \mathrm{Rad}(U) \ll U$ according to [14, Proposition 7]. We consider two possible cases for $\mathrm{Rad}(U)$.

Case 1: If $\mathrm{Rad}(U)$ is a simple subsemimodule of U, since $\mathrm{Rad}(U) = \cap_{i\in I} U_i$, where U_i is the maximal subsemimodules of U, for all $i \in I$, we choose $N = \cap_{i\in I-\{1\}} U_i$. Then $\{U_1, N, \mathrm{Rad}(U)\}$ is a triangle in the graph $\Gamma(U)$.

Case 2: If $\mathrm{Rad}(U)$ is not a simple subsemimodule of U, then there exists a non-trivial subsemimodule X of U which $X \subset \mathrm{Rad}(U)$. By [14, Proposition 7], $\mathrm{Rad}(U) \ll U$. And by [14, Proposition 3], $X \ll U$. Thus for each vertex Y of $\Gamma(U), \{X, Y, \mathrm{Rad}(U)\}$ is a triangle in $\Gamma(U)$. □

Example 3.1. Let p be a prime number. Consider $\mathbb{Z}_{p^3}$ as a $\mathbb{Z}$-semimodule. There are only two non-trivial subsemimodules $p\mathbb{Z}_{p^3}$ and $p^2\mathbb{Z}_{p^3}$ such that $p\mathbb{Z}_{p^3} + p^2\mathbb{Z}_{p^3} \neq \mathbb{Z}_{p^3}$. Hence $p\mathbb{Z}_{p^3}$ and $p^2\mathbb{Z}_{p^3}$ are small. So, $p\mathbb{Z}_{p^3} \cap p^2\mathbb{Z}_{p^3} \ll \mathbb{Z}_{p^3}$. Thus $\Gamma\left(\mathbb{Z}_{p^3}\right) \cong P_2$.

Definition 3.6. [7] A graph is said to be planar, if it has a drawing in a plane without crossings.

Theorem 3.7. [7, Theorem 10.30] A graph is planar if it contains no subdivision of either K_5 or $K_{3,3}$.

Proposition 3.8. Suppose that the intersection of every pair of non-small subsemimodules of U is a nonsmall subsemimodule. If $|\mathbb{S}(U)| = 1$ or $|\mathbb{S}(U)| = 2$, then $\Gamma(U)$ is a planar graph.

Proof. Assume that $|\mathbb{S}(\mathsf{U})| = 1$. Then $\Gamma(\mathsf{U})$ contains a vertex I which is adjacent to each other vertex. According to assumption, if $\mathsf{N} \neq \mathsf{I}$ and $\mathsf{K} \neq \mathsf{I}$ are two distinct vertices of $\Gamma(\mathsf{U})$, then N and K are not adjacent vertices. Thus $\Gamma(\mathsf{U})$ is a star with center I. So $\Gamma(\mathsf{U})$ is a planar graph. Now, if $|\mathbb{S}(\mathsf{U})| = 2$, then $\Gamma(\mathsf{U})$ does not contain $\mathsf{K}_{3,3}$ or K_5 and by Theorem 3.7, thus $\Gamma(\mathsf{U})$ is a planar graph. □

The next proposition exhibits that the planarity of $\Gamma(\mathsf{U})$ limits the order of $\mathbb{S}(\mathsf{U})$.

Proposition 3.9. If $|\mathbb{S}(\mathsf{U})| \geqslant 3$, then $\Gamma(\mathsf{U})$ is not a planar graph.

Proof. Suppose that $|\mathbb{S}(\mathsf{U})| \geqslant 3$. Then U has at least three different nonzero small subsemimodules, say M, N and P. Obviously, any one of the vertices $\mathsf{M}+\mathsf{N}, \mathsf{N}+\mathsf{P}$ and $\mathsf{M}+\mathsf{P}$ are non-zero subsemimodule and adjacent to each of subsemimodules M, N and P in $\Gamma(\mathsf{U})$. Therefore, $\Gamma(\mathsf{U})$ contains a complete graph K_5 such as the subgraph induced on the set $\{\mathsf{M}, \mathsf{N}, \mathsf{P}, \mathsf{M}+\mathsf{N}, \mathsf{N}+\mathsf{P}\}$. Hence, by the definition of planar graph in Theorem 3.7, $\Gamma(\mathsf{U})$ is not a planar graph. □

Finally, we give the following main result.

Proposition 3.10. Let $\mathsf{U} = \mathsf{V} \oplus W$ be a finitely generated subtractive R-semimodule such that $\mathrm{Rad}(\mathsf{V}) \neq (0)$ and $\mathrm{Rad}(W) \neq (0)$. Then the graph $\Gamma(\mathsf{U})$ is not planar.

Proof. Let $\mathsf{U} = \mathsf{V} \oplus W$. Then $\mathrm{Rad}(\mathsf{U}) = \mathrm{Rad}(\mathsf{V}) \oplus \mathrm{Rad}(W)$ and $\mathrm{Rad}(\mathsf{V}) \cap \mathrm{Rad}(W) = (0) \ll \mathsf{U}$. Also $\mathrm{Rad}(\mathsf{V}) \subseteq \mathsf{V}$ and $\mathsf{V} \cap \mathrm{Rad}(W) \subseteq \mathsf{V} \cap W = (0)$, and so the Semimodularity Law (see Proposition 2.10), implies that $\mathsf{V} \cap \mathrm{Rad}(\mathsf{U}) = \mathrm{Rad}(\mathsf{V})$ and similarly, $W \cap \mathrm{Rad}(\mathsf{U}) = \mathrm{Rad}(W)$. Moreover, $\mathsf{V} \cap \mathrm{Rad}(\mathsf{V}) = \mathrm{Rad}(\mathsf{V}) \ll \mathsf{U}$, $W \cap \mathrm{Rad}(W) = \mathrm{Rad}(W) \ll \mathsf{U}$, $\mathsf{V} \cap \mathrm{Rad}(\mathsf{U}) = \mathrm{Rad}(\mathsf{V}) \ll \mathsf{U}$, $W \cap \mathrm{Rad}(\mathsf{U}) = \mathrm{Rad}(W) \ll \mathsf{U}$, $\mathrm{Rad}(\mathsf{V}) \cap \mathrm{Rad}(\mathsf{U}) = \mathrm{Rad}(\mathsf{V}) \ll \mathsf{U}$ and $\mathrm{Rad}(W) \cap \mathrm{Rad}(\mathsf{U}) = \mathrm{Rad}(W) \ll \mathsf{U}$. Hence $\mathsf{V}, W, \mathrm{Rad}(\mathsf{V}), \mathrm{Rad}(W)$ and $\mathrm{Rad}(\mathsf{U})$ are adjacent vertices in the graph $\Gamma(\mathsf{U})$. So, the set $\{\mathsf{V}, W, \mathrm{Rad}(\mathsf{V}), \mathrm{Rad}(W), \mathrm{Rad}(\mathsf{U})\}$ induces a complete subgraph K_5 in $\Gamma(\mathsf{U})$. Thus, according to Theorem 3.7, $\Gamma(\mathsf{U})$ is not a planar graph. □

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